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A DIVERGENT TEICHMÜLLER GEODESIC WITH UNIQUELY ERGODIC VERTICAL FOLIATION

BY

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ABSTRACT

We construct an example of a quadratic differential whose vertical foliation is uniquely ergodic and such that the Teichmüller geodesic determined by the quadratic differential diverges in the moduli space of Riemann surfaces.

1. Introduction and statement of theorem

We let S be a closed surface of genus $g \geq 2$ and T_g the *Teichmüller space* of S equipped with the Teichmüller metric. Let Map $= \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$ denote the *mapping class group* of S. It acts on T_g by isometries with quotient space $\mathcal{M}_g$, the *moduli space* of surfaces of genus g. It is well-known that $\mathcal{M}_g$ is not compact; one leaves compact sets of $\mathcal{M}_g$ by finding curves whose lengths in the hyperbolic metric on the surfaces go to zero. For any Riemann surface $X \in T_g$ a *holomorphic quadratic differential* $q = q(z)dz^2$ assigns to each uniformizing parameter z a holomorphic function $q(z)$ such that $q(z)dz^2$ is invariant under

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holomorphic change of coordinates. Away from the zeroes of q there are natural coordinates so that $q(z) \equiv 1$, so that q defines a metric $|dz^2|$ which is locally Euclidean away from the zeroes of q. A *saddle connection* γ is a geodesic in this metric joining a pair of (not necessarily distinct) zeroes that has no zeroes in its interior. It is represented by a straight line in the natural Euclidean metric and determines a *holonomy vector* with horizontal and vertical components which we denote by $\lambda(\gamma)$ and $v(\gamma)$ respectively.

Associated to q are the *horizontal* and *vertical* trajectories. These are the arcs along which $q(z)dz^2 > 0$ and $q(z)dz^2 < 0$. For each $t \in \mathbb{R}$ one can define a new quadratic differential $g_t(q)$ on a new Riemann surface X_t by expanding along the *horizontal* trajectories of q by a factor of e^t and contracting along the vertical trajectories by a factor of e^{-t}. The family of X_t forms a *Teichmüller geodesic* through X whose projection to $\mathcal{M}_g$ defines a geodesic in M_g. The underlying map $X \to X_t$ of Riemann surfaces is called a *Teichmüller map.* In the appropriate natural coordinates $z = x + \sqrt{-1}y$ on X away from the zeroes of q, and natural coordinates $z_t = x_t + \sqrt{-1}y_t$ on X_t away from the zeroes of $g_t(q)$, the map is given by

$$x_t = e^t x, \quad y_t = e^{-t} y.$$

One may study the dynamics of the family of vertical trajectories of q which defines the *vertical foliation* F_q. An important notion in topological dynamics is that of minimality. The foliation F_q is *minimal* if the full orbit of every leaf is dense. It is a standard fact [St] that if there are no saddle connections with zero horizontal holonomy, then the vertical foliation F_q is minimal.

For minimal foliations it is natural to study their ergodic behavior. A foliation is *uniquely ergodic* if every leaf is uniformly distributed on the surface. Equivalently, the foliation is uniquely ergodic if there is a unique, up to scalar multiplication, measure transverse to F_q invariant under holonomy along leaves of F_q. (To avoid trivialities, we assume these measures are supported on the complement of the singularities.) An interesting phenomenon found by [S], [V1], [K], [KN] is the existence of minimal foliations that are not uniquely ergodic. The last two examples were in the context of interval exchange transformations, but a suspension of an interval exchange transformation yields a (oriented) measured foliation.

In [M] a connection was made between the dynamics of the foliation F_q and the dynamics of the corresponding geodesic X_t determined by q in $\mathcal{M}_g$. Another connection was made in [B]. It was shown that if the foliation is minimal, but not uniquely ergodic, then the geodesic *diverges* in $\mathcal{M}_g$. This means that it eventually leaves every compact set. The question arises if this condition is necessary; does X_t divergent imply F_q is nonuniquely ergodic? We show the answer is negative.

THEOREM 1: *There exist quadratic differentials q with uniquely ergodic F_q such that the Teichmuller geodesic X_t diverges in $\mathcal{M}_g$.*

We will construct an example in genus two, although our method will produce examples in any genus. That such examples should exist was already suggested by Kerckhoff to the second author (oral communication). We consider the "zippered rectangle" construction of Veech [V2]. This is a way of defining an Abelian differential on a Riemann surface such that the first return map of the vertical trajectories of the Abelian differential to a horizontal interval is a given interval exchange transformation.

We then recall the notion of Rauzy induction ([R]). This is a procedure which by taking the first return map on a subinterval of the horizontal interval gives a new interval exchange and also gives a new zippered rectangle. Associated to Rauzy induction is a graph of permutations of interval exchanges. The point of this construction in our context is that successive applications of Rauzy induction give a discrete set of points along the Teichmüller geodesic defined by the zippered rectangles.

We then specialize to interval exchanges on four intervals and explicitly find the graph of permutations. We produce an infinite path in this graph that allows us to explicitly compute the lengths and heights of the zippered rectangles given by Rauzy induction. The interval exchange whose sequence of Rauzy inductions give this path has the desired properties. Namely, it is uniquely ergodic, and any Abelian differential formed by the zippered rectangle construction yields a divergent geodesic.

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2. Interval exchanges and zippered rectangles

Let $\lambda \in \mathbb{R}^m_+$ be a vector of positive lengths and π an irreducible permutation on $\{1, \dots, m\}$ for some $m \geq 2$. Recall irreducibility means $\pi\{1, \dots, k\} \neq \{1, \dots, k\}$ for $1 \leq k < m$. Associated to the pair (λ, π) is an interval exchange map T defined as follows. Let $I_j = [\beta_{j-1}, \beta_j)$ where $\beta_0 = 0$ and $\beta_j = \lambda_1 + \cdots + \lambda_j$ for

$j = 1, \dots, m$. Then T is the map defined on $I = \cup I_j$ by the formula: for $x \in I_j$

$$T(x) = x - \sum_{i<j} \lambda_i + \sum_{\pi i < \pi j} \lambda_{\pi j}.$$

We recall a construction in [V2] of "zippered rectangles" to suspend T. Let $h, a \in \mathbb{R}^m$ be vectors satisfying the inequalities

$$h_j > 0 \qquad (1 \le j \le m), \tag{1}$$

$$0 < a_j \le \min(h_j, h_{j+1}) \qquad (1 \le j < m, j \ne \pi^{-1}m), \tag{2}$$

$$0 < a_{\pi^{-1}m} \le h_{\pi^{-1}m+1}, \tag{3}$$

$$-h_{\pi^{-1}m} \le a_m \le h_m, \tag{4}$$

and the system of linear equations

$$h_j - a_j = h_{\sigma j+1} - a_{\sigma j} \quad (0 \le j \le m), \tag{5}$$

where $a_0 = h_0 = h_{m+1} = 0$ by convention and σ is the permutation on $\{0, 1, \dots, m\}$ defined by

$$\sigma j = \begin{cases} \pi^{-1}1 - 1, & j = 0, \\ m, & j = \pi^{-1}m, \\ \pi^{-1}(\pi j + 1) - 1, & \text{otherwise.} \end{cases} \tag{6}$$

The surface $M(\pi, \lambda, h, a)$ is obtained by glueing together m rectangles along their boundaries in the following manner. Let R_j be the rectangle in $\mathbb{C}$ with base I_j on the real axis and height h_j. There are three cases depending on the sign of a_m and in each case there will be three sets of identifications, all of which are the form $z \to z + c$.

First, consider the case $a_m = 0$. The first set of identifications is

(1) the top of R_j is glued to the interval TI_j at the base for $j = 1, \dots, m$.

To describe the remaining identifications we shall use the notation

$$R_j^+[a, b] \sim R_k^-[c, d]$$

to mean the right side of R_j between heights a and b is glued to the left side of R_k between heights c and d. (For this to be well-defined, we must have $b - a = d - c$.) The remaining identifications in the case $a_m = 0$ are

(2) $R_j^+[0, a_j] \sim R_{j+1}^-[0, a_j]$ for $j = 1, \dots, m-1$, and

(3) $R_j^+[a_j, h_j] \sim R_{j+1}^-[a_{\sigma j}, h_{\sigma j+1}]$ for $j = 1, \dots, m$.

Now consider the case $a_m > 0$. Let $j = \pi^{-1}m$. All identifications remain the same except the jth in (2), which is replaced with

$$R^+_{\pi^{-1}m}[0, h_j] \sim R^-_{\pi^{-1}m+1}[0, h_j] \quad \text{and}$$
$$R^+_m[0, a_m] \sim R^-_{\pi^{-1}m+1}[h_j, a_j].$$

Similarly, in the case $a_m < 0$ all identifications remain the same as in the first case except the mth in (3), which is replaced with

$$R^+_{\pi^{-1}m}[a_{\pi^{-1}m}, h_{\pi^{-1}m}] \sim R^-_{\sigma m+1}[a_{\sigma m}, h_{\sigma m+1} - h_m] \quad \text{and}$$
$$R^+_m[0, h_m] \sim R^-_{\sigma m+1}[h_{\sigma m+1} - h_m, h_{\sigma m+1}].$$

The collection of glued rectangles $M = M(\pi, \lambda, h, a)$ is called the *zippered rectangle* associated to (π, λ, h, a). Since the glueing maps are of the form $z \to z + c$, M carries the structure of a Riemann surface and the 1-form dz induces an Abelian differential $\omega = \omega(\pi, \lambda, h, a)$ on M. The interval exchange T is the first return map to I under the flow in the vertical direction generated by the vector field $\partial/\partial y$.

Note if each cycle of σ contains at least two elements in $\{1, \dots, m-1\}$, then ω has a zero at $(0,0)$ and at $\lambda_1 + \cdots + \lambda_j + ia_j$ for $j = 1, \dots, m$. In particular, each R_j has at most one zero on each of its vertical sides. More specifically, the left side always contains a zero while the right side contains a zero except when $j = m$, $a_m < 0$ or when $j = \pi^{-1}m$, $a_m > 0$.

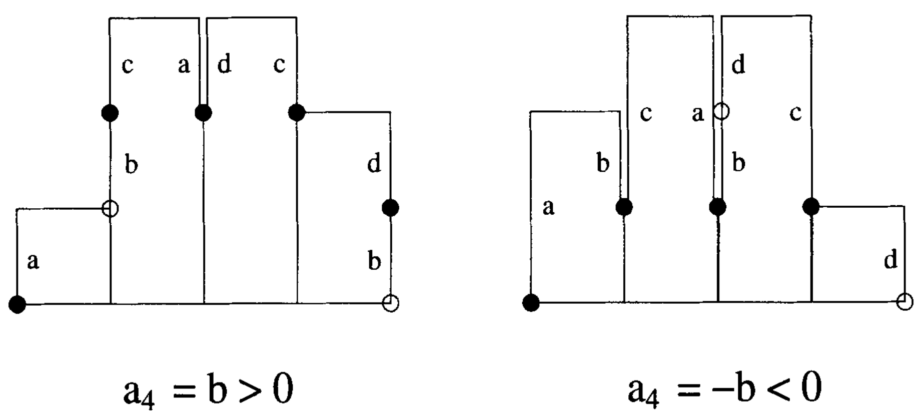


Figure 1. Zippered rectangles in Example 1. On the left $a_1 = a + b$, $a_3 = b + d$ and $a_4 = b$; on the right $a_1 = a - b$, $a_3 = d$ and $a_4 = -b$. In either case, $a_0 = 0$ and $a_2 = h_2 - a$.

Example 1: Let $m = 4$, $\pi j = 5 - j$ and $\lambda \in \mathbb{R}^4_+$. Then (π, λ, h, a) is a zippered rectangle with $h = (1,3,3,2)$ and $a = (2,2,2,1)$ or with $h = (2,3,3,1)$ and $a = (1,1,1,-1)$, where $\sigma j = j - 2 \pmod 5$. See Figure 1.

3. Rauzy induction

Let $\mathcal{A}$ denote an alphabet with $m \geq 2$ elements. A *marked interval exchange* is a pair (T, ν_0) where T is an interval exchange on m intervals and ν_0 a bijection from $\{1, \ldots, m\}$ to $\mathcal{A}$. We think of the intervals of T as being marked from left to right with names $\nu_0(1), \ldots, \nu_0(m)$. If T is a (λ, π) interval exchange, then the names of the image intervals from left to right are given by $\nu_1(1), \ldots, \nu_1(m)$ where

$$\nu_1 = \nu_0 \circ \pi^{-1}.$$

An alternative way to represent a marked interval exchange is via a triple (λ, ν_0, ν_1) where $\lambda\colon \mathcal{A} \to \mathbb{R}_+$ is the function given by

$$\lambda(\nu_0(j)) = \lambda_j \quad \text{for } j = 1, \ldots, m.$$

The function λ and the vector $\lambda \in \mathbb{R}^m_+$ are represented by the same symbol so that $\lambda(\alpha)$ is the length of the interval marked α, while λ_j is the length of the jth interval from left to right. We shall also refer to the pair (ν_0, ν_1) as a *marked permutation.* In what follows, we fix $\mathcal{A} = \{1, \ldots, m\}$.

We define *Rauzy induction* on the space of marked interval exchange transformations. First for λ a vector let $|\lambda|$ denote the sum of the lengths of the components. Now given a marked interval exchange (λ, ν_0, ν_1), consider the first return map found by inducing on the longer of the two subintervals $[0, |\lambda| - \lambda(\nu_i(m)))$ where $i \in 0, 1$. If

$$\text{Case (a)} \quad \lambda(\nu_1(m)) > \lambda(\nu_0(m))$$

then the interval marked $\nu_1(m)$ is shortened. The new marked permutation is (ν_0', ν_1') where $\nu_1' = \nu_1$ and ν_0' is the ordering obtained from ν_0 by inserting $\nu_0(m)$ in the position immediately after $\nu_1(m)$ and moving forward one place the names of the intervals appearing after $\nu_1(m)$.

Likewise, if

$$\text{Case (b)} \quad \lambda(\nu_0(m)) > \lambda(\nu_1(m))$$

then the interval marked $\nu_0(m)$ is shortened, $\nu_0' = \nu_0$ and ν_1' is the ordering obtained from ν_1 by inserting $\nu_1(m)$ in the position immediately after $\nu_0(m)$ and moving forward one place the names of the intervals appearing after $\nu_0(m)$. We ignore the case $\lambda(\nu_0(m)) = \lambda(\nu_1(m))$.

The result of Rauzy induction is a new marked interval exchange $(\lambda', \nu_0', \nu_1')$. In the first case the lengths are related by

$$\lambda = A\lambda', \tag{7}$$

where A is the elementary matrix whose only nonzero off-diagonal entry is a 1 in the $(\nu_1(m), \nu_0(m))$ position. In the second case the matrix is the transpose of the above matrix so that it has a 1 in the $(\nu_0(m), \nu_1(m))$ position. We can rephrase the statement about the new marked permutation as follows. The new marked permutation is $a(\nu_0, \nu_1)$ in the first case and $b(\nu_0, \nu_1)$ in the second where a, b are the bijections on the set of marked permutations defined by

$$a(\nu_0, \nu_1) = (\nu_0 \circ c_k, \nu_1), \quad k = \nu_1^{-1}(m), \tag{8}$$

$$b(\nu_0, \nu_1) = (\nu_0, \nu_1 \circ c_l), \quad l = \nu_0^{-1}(m), \tag{9}$$

and

$$c_k(i) = \begin{cases} i, & i \le k, \\ m, & i = k+1, \\ i-1, & k+1 < i \le m. \end{cases} \tag{10}$$

(The context will make it clear whether a is an operation on the set of marked permutations or a vector of lengths in $\mathbb{R}^m$.)

By repeating this procedure, we obtain a sequence of elementary matrices $(A_k)_{k\ge1}$ and length vectors $(\lambda_k)_{k\ge1}$ satisfying

$$\lambda = A_1 \cdots A_k \lambda_k.$$

The sequence $A_1 \cdots A_k \cdots$ is called the *expansion* of λ. The corresponding sequence of marked permutations is a path in the extended Rauzy diagram. This is a directed graph whose vertex set is the extended Rauzy class $R(\nu_0, \nu_1)$ consisting of all marked permutations obtainable by applying the operations a and b to (ν_0, ν_1). There is a directed edge from x to y if and only if either $y = ax$ or $y = bx$. The edges in the Rauzy diagram are labeled by a or b as the case may be. A directed path starting at (ν_0, ν_1) is uniquely represented by a word in the alphabet $\{a, b\}$. If w is a word in $\{a, b\}$, let $\langle x; w\rangle$ denote the path starting at x obtained by following the letters of w from left to right. Let $[x; w]$ denote the corresponding product of elementary matrices. If the path ends at y and w' is another word, then $[x; ww'] = [x; w][y; w']$ where ww' denotes the concatenation of the words w and w'.

3.1 Rauzy induction with heights. Rauzy induction also gives a transformation on the space of zippered rectangles. Let $M(\pi, \lambda, h, a)$ be a zippered rectangle. Let

$$h' = A^t h \tag{11}$$

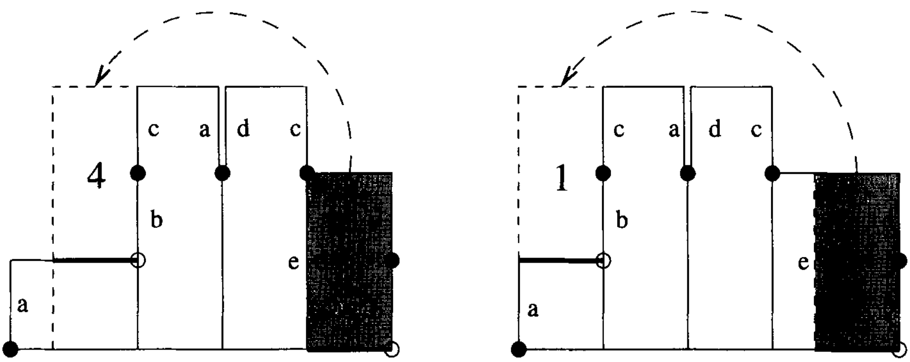
and define, if $\lambda_m < \lambda_{\pi^{-1}m}$ (edge a),

$$a'_j = \begin{cases} a_j, & j < \pi^{-1}m, \\ h_{\pi^{-1}m} + a_{m-1}, & j = \pi^{-1}m, \\ a_{j-1}, & \pi^{-1}m < j \le m, \end{cases} \tag{12}$$

and if $\lambda_m > \lambda_{\pi^{-1}m}$ (edge b),

$$a'_j = \begin{cases} a_j, & j < m, \\ -(h_{\pi^{-1}m} - a_{\pi^{-1}m-1}), & j = m. \end{cases} \tag{13}$$

Then the Riemann surface $M(\pi, \lambda, h, a)$ is biholomorphically equivalent to $M(\pi', \lambda', h', a')$ and the Abelian differential $\omega(\pi, \lambda, h, a)$ is equivalent to the Abelian differential $\omega(\pi', \lambda', h', a')$. In the first case, the last rectangle is stacked on top (as far right as possible) of a piece of the one that goes to the last. In the second case a piece of the last rectangle is stacked on top of the one that goes to the last. See Figure 2. In particular, the last rectangle of the new set of zippered rectangles has a zero on its right side if the corresponding edge is labeled with an 'a' (the first case).



(a) $\lambda(1) > \lambda(4)$ (b) $\lambda(4) > \lambda(1)$

Figure 2. Rauzy induction with (π, λ, h, a) as in Example 1. The solid line indicates where a piece of the last rectangle is stacked on top of the first. Note that the name of the rectangle with a new height is 4 in the first case and 1 in the second.

4. Interval exchanges with four intervals

For the rest of the paper we will consider interval exchanges on four intervals. Let $\pi_0 = (\nu_0, \nu_1)$ where $\nu_0(j) = j$ and $\nu_1(j) = 5 - j$. The extended Rauzy diagram $R(\pi_0)$ is shown in Figure 3.

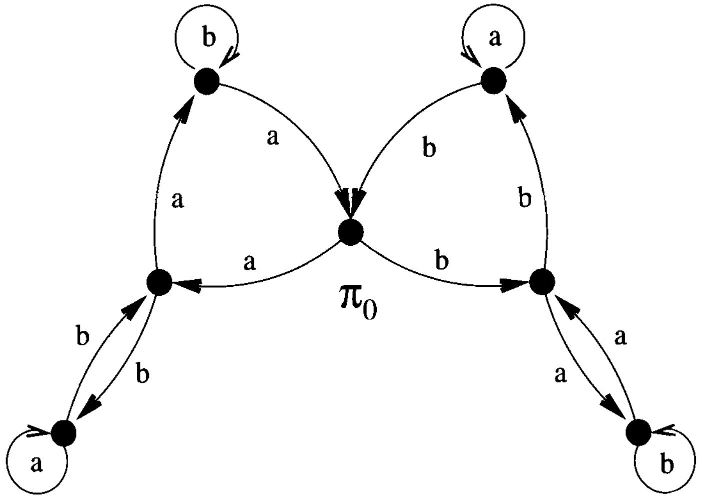


Figure 3. Rauzy diagram for the marked permutation π_0.

Now for each n we form a pair of paths

$$\langle \pi_0; bab^n ab^2 \rangle$$

and

$$\langle \pi_0, aba^n ba^2 \rangle$$

that begin and end at π_0. We form the corresponding product of elementary matrices $A_n = [\pi_0; bab^n ab^2]$ and $B_n = [\pi_0; aba^n ba^2]$ and set $C_n = A_n B_n$. One computes C_n as follows. At each stage on the path, the marked permutation (ν_0', ν_1') is determined from the previous marked permutation (ν_0, ν_1) by (8) and (9). Each marked permutation determines an elementary matrix as in (7) and the discussion that follows. The matrix C_n is the product of these elementary matrices. It is then easily verified that the entries of C_n are polynomials in n whose leading terms are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2n & 2n & 1 \\ n & n^2 & n^2 & 0 \\ 2 & n & n & 2 \end{bmatrix}.$$

For each n, set

$$H_n = \prod_{j=1}^{n} C_j.$$

For any $\lambda \in H_n \mathbb{R}^4_+$, let λ_n be such that

$$\lambda = H_n \lambda_n.$$

Then successive applications of Rauzy induction on the interval exchange (λ, π_0) produces the interval exchange (λ_n, π_0). Now let Σ be the standard simplex in $\mathbb{R}^4$ and regard each C_j as a projective linear transformation from Σ to itself.

Proposition 1: *The sets $\Sigma_n = H_n \Sigma$ are a decreasing sequence with infinite intersection a single point denoted by λ_0. The interval exchange (λ_0, π_0) is uniquely ergodic.*

In order to prove the proposition we use the Hilbert metric on Σ. We will prove a general proposition stronger than what is needed. For any $m \times m$ matrix L with all positive entries L_{ij} set

$$\delta(L) = \min_{i,j,r,s} \frac{L_{ir} L_{js}}{L_{is} L_{jr}}.$$

Proposition 2: *Let Σ be the open standard simplex in $\mathbb{R}^m$. Let $A_j \colon \Sigma \to \Sigma$ be a sequence of projective linear transformations defined by positive matrices with*

$$\prod_{j=1}^{\infty} (1 - \delta(A_j)) = 0.$$

Then for any sequence U_j of matrices with nonnegative entries,

$$\bigcap_{n=1}^{\infty} A_1 U_1 A_2 U_2 \cdots A_n U_n \Sigma$$

is a single point.

Proof: For $x, y \in \Sigma$ define

$$\Gamma(x,y) = \inf_{x_i, y_j \neq 0} \frac{x_j y_i}{x_i y_j}$$

and the Hilbert metric

$$d(x,y) = -\log \Gamma(x,y).$$

By Lemma 15.1(iv) of [F], any matrix U with nonnegative entries satisfies

$$d(Ux, Uy) \leq d(x, y)$$

so that it is enough to show that

$$d(Lx, Ly) \leq (1 - \delta(L))d(x, y).$$

In fact, it is enough to show this inequality for $\epsilon := d(x, y)$ arbitrarily small, since the Hilbert metric is a path metric, i.e. for any z on (the) geodesic joining x and y, $d(x, y) = d(x, z) + d(z, y)$. (This follows readily from the definition or by observing that the metric restricts to a Riemannian metric on any linear segment.)

By Lemma 15.2 of [F] we have

$$\Gamma(Lx, Ly) \geq \frac{\delta + \Gamma(x, y)}{1 + \delta\Gamma(x, y)}.$$

Writing $\Gamma(x, y) = e^{-\epsilon} = 1 - \epsilon + o(\epsilon)$, we see that the right hand side of the above expression is of the form

$$\frac{\delta + 1 - \epsilon + o(\epsilon)}{1 + \delta - \delta\epsilon + o(\epsilon)},$$

and so

$$\Gamma(Lx, Ly) \geq 1 - \frac{1 - \delta}{1 + \delta}\epsilon + o(\epsilon),$$

implying that

$$d(Lx, Ly) \leq \frac{(1 - \delta)}{(1 + \delta)}d(x, y) + o(d(x, y)),$$

which gives the desired estimate. ∎

We now turn to the proof of Proposition 1. From the form of the leading terms of the matrix C_n we compute that the entries of the matrix $C_n C_{n+1}$ are all positive polynomials in n whose leading terms are

$$\begin{bmatrix} n & n^2 & n^2 & 4 \\ 2n^2 & 2n^3 & 2n^3 & 2n \\ n^3 & n^4 & n^4 & n^2 \\ n^2 & n^3 & n^3 & n \end{bmatrix}.$$

We see immediately that there is an integer $p = p(i, j) \in [-2, 2]$ such that the entries of $C_n C_{n+1}$ satisfy

$$1/4 < n^p \frac{a_{i,r}}{a_{j,r}} < 4$$

for any r, so that

$$\delta(C_n C_{n+1}) = 1/16 + O(1/n).$$

Proposition 2 then implies that

$$\mathrm{diam}(H_n(\Sigma)) \to 0$$

as $n \to \infty$. Now it is a standard fact [K], [KE] that if $\lambda_0 = \cap H_n\Sigma$ is minimal, it is uniquely ergodic.

We now show that (λ_0, π_0) is minimal. Fix a unit area Abelian differential $\omega_0 = \omega_0(\pi_0, \lambda_0, h_0, a_0)$. (This means that the first return map of the vertical leaves to a horizontal segment defines the interval exchange (λ_0, π_0) and ω_0 defines a zippered rectangle with height vector $h_0 = (h_0(1), \ldots, h_0(4))$.)

If (λ_0, π_0) is not minimal then ω_0 has a vertical saddle connection γ. This saddle connection intersects the horizontal segment corresponding to λ_0 a finite number of times. Choose n large enough so that γ does not intersect the interior of the horizontal segment corresponding to λ_n. This means that there is some rectangle in the "zippered rectangles" determined by λ_n that has γ as part of its vertical side. In that case there would be a pair of zeros on this side, which is a contradiction to the following property of the zippered rectangle construction: *each side of the new rectangles contains at most one zero.*

5. Proof of Theorem 1

We show that $\omega_0 = \omega_0(\pi_0, \lambda_0, h_0, a_0)$ determines a divergent Teichmüller geodesic X_t. To do that we need to show that for all sufficiently large times t along the geodesic X_t there is a simple closed curve γ_t which is short in the flat metric defined by $g_t(\omega_0)$, since this guarantees a short curve in the hyperbolic metric.

The word $bab^n ab^2 aba^n ba^2$ that is used to form the matrix $C_n = A_n B_n$ ends with a. Recall that this means that the zippered rectangle for each interval exchange (λ_n, π_0) has a zero on the right side of the fourth rectangle. Thus there is a saddle connection γ_n such that the horizontal component $\lambda(\gamma_n)$ of its holonomy is equal to $\lambda_n(4)$, and such that the vertical component $v(\gamma_n)$ of its holonomy is at most $h_n(4)$. We will show that this saddle connection becomes short in some time interval and that these time intervals cover (T, ∞) for large enough T. Since ω_0 has only one singularity, this saddle connection will always be a closed curve.

We first give a bound for $\lambda_n(4)h_n(4)$. By (7) the length vectors satisfy the equation

$$\lambda_n = C_{n+1}C_{n+2}\lambda_{n+2}$$

and by (11) the height vector satisfies

$$h_n = (C_{n-1}C_n)^t h_{n-2}. \tag{14}$$

The form of the leading terms of these matrices insures that for $i \neq 3$,

$$\frac{\lambda_n(i)}{\lambda_n(3)} = O(1/n) \tag{15}$$

and

$$\frac{h_n(i)}{h_n(3)} = O(1). \tag{16}$$

Since

$$\sum_{i=1}^{4} \lambda_n(i)h_n(i) = 1,$$

the above estimates show that

$$\lambda_n(3)h_n(3) = 1 + O(1/n). \tag{17}$$

Moreover, the form of the leading terms of the matrices also gives

$$\frac{h_n(4)}{h_n(3)} = O(1/n^2),$$

which together with (15) and (17) gives

$$\lambda_n(4)h_n(4) = O(1/n^3). \tag{18}$$

Now define s_n by

$$e^{-s_n}v(\gamma_n) = 1/\log n \tag{19}$$

and t_n by

$$e^{t_n}\lambda(\gamma_n) = 1/\log n. \tag{20}$$

Now the flow g_t contracts vertical holonomy by e^{-t} and expands horizontal holonomy by e^t. By a slight abuse of notation we denote by $g_t(\gamma_n)$ the saddle connection γ_n with respect to the Abelian differential $g_t(\omega_0)$.

These times are chosen so that for any $t \in (s_n, t_n)$

$$v(g_t(\gamma_n)) = e^{-t}v(\gamma_n) \leq 1/\log n$$

and

$$\lambda(g_t(\gamma_n)) = e^t\lambda(\gamma_n) \leq 1/\log n,$$

which implies that the length of $g_t(\gamma_n)$ is bounded by $2/\log n$ for any t in this interval.

If we can show that the intervals (s_n, t_n) and (s_{n+1}, t_{n+1}) overlap, then we would be done, since this would imply that at all times there is a short saddle connection. By (18), (19), (20) and the fact that $v(\gamma_n) \leq h_n(4)$, it is implied that

$$t_n - s_n \geq 3\log n + O(\log\log n). \tag{21}$$

Now the form of the leading terms of the matrix $C_{n+1}C_{n+2}$ implies that

$$\frac{\lambda_n(3)}{\lambda_n(4)} \asymp O(n).$$

(Here, $A \asymp O(n)$ means $A = O(n)$ and $1/A = O(1/n)$.) Using (15) and the form of the leading terms of the matrix C_{n+1} we have

$$\frac{\lambda_n(3)}{\lambda_{n+1}(3)} = n^2 + O(1/n),$$

so that

$$\frac{\lambda_{n+1}(4)}{\lambda_n(4)} \asymp O(1/n^2). \tag{22}$$

Now inequality (22) implies that

$$t_{n+1} - t_n = 2\log n + O(1)$$

and, together with (21), it implies that for large n,

$$(s_n, t_n) \cap (s_{n+1}, t_{n+1}) \neq \emptyset.$$

This finishes the proof.

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